18th Conference on Water Distribution System Analysis, WDSA 2016

Sensitivity Analysis of Topological Subgraphs within Water Distribution Systems

Jochen Deuerlein[a]*, Olivier Piller[b], Sylvan Elhay[c], Angus Simpson[d]

[a]3S Consult GmbH, Albtalstrasse 13, 76137 Karlsruhe, Germany
[b]Irstea, UR ETBX, Water Department, Bordeaux regional centre, Cestas F-33612, France
[c]School of Computer Science, University of Adelaide, SA 5005,
[d]School of Civil, Environmental and Mining Engineering, University of Adelaide, Adelaide SA 5005, Australia

Abstract

Sensitivity analysis of the actual hydraulic state of water distribution systems is a valuable tool with number of applications in hydraulic systems analysis. Sensitivity matrices include the information of the response of the hydraulic state variables (flows, pressures) to changes in model parameters (e.g. demands, roughness, control parameters) for a specific hydraulic state of the system. For calculation, there exists in addition to finite difference approximations also exact solutions that include the inversion of the system matrix (the Schur Complement of the Jacobian of the hydraulic network equations). In combination with hydraulic network simulations, the factorization (for example Cholesky matrix decomposition) of the matrix that has already been done by the hydraulic solver in the computation process can be used for the efficient calculation of the inverse matrix. However, for large real-world networks the sensitivity calculation is a time and memory consuming process because the inverse of the system matrix of a connected network has no zero elements. In this paper a new method is presented that allows for the exact calculation of sensitivities of a particular subgraph of interest, the topological minor or supergraph within a water distribution system network graph. Supernodes are the most important nodes in terms of connectivity redundancy within the network graph. Superlinks replace all pipes (links) in series between two supernodes. It will be shown that the sensitivities that are calculated for the subgraph deliver exactly the same results as the inversion of the entire system matrix reduced to supernodes. This paper focuses on the derivation of the equations for the reduced system matrix inversion of the topological subgraph. In addition, the paper includes the proof of equivalence of the matrix inverses for the topological minor subgraph. This inverse represents the fundamental sensitivities of nodal heads and pipe flows with respect to nodal demands in demand driven analysis. The results presented can be extended to other sensitivities, since the matrix inverse in question is included in all other derived parameter sensitivities.

Keywords: Water distribution systems, sensitivity analysis, topological minor, matrix inversion;

* Corresponding author. Tel.: +49 721 33503360; fax: +49 721 335033130.
E-mail address: deuerlein@3sconsult.de

1. Introduction

Sensitivity analysis plays an important role in hydraulic system engineering. Sensitivities are first order estimates of the change of the state variables (flows, heads) with respect to different kind of parameter changes. Consequently, sensitivity analysis can be used for example for hydraulic model calibration [1]. The calculation of the sensitivities is normally a computationally costly procedure since it involves the inversion of the Schur Complement $\mathbf{J} = \mathbf{A}^T\mathbf{F}^{-1}\mathbf{A}$ of the Jacobian matrix of the system equations of the Global Gradient Algorithm [2]. In this paper a new method is proposed that enables the calculation of the analytically exact sensitivities for a smaller subset of nodes. It will be proven that these sensitivities are equivalent with those that results from the inversion of the full system matrix. In terms of time saving, this is a very important result.

In a previous paper the Graph Matrix Partitioning Algorithm (GMPA) was introduced [3]. It was shown that the hydraulic steady-state calculation of large complex pipe networks can be split into a local and a global solution. Whereas the global solution includes solving a nonlinear system of equations the local solution consists of simple linear calculations. The basic idea of this method was the observation that supply networks commonly include a number of tree-like subgraphs (e.g. the large number of subsystems representing the secondary distribution and house connection pipes) that can be treated separately in a more efficient manner. The nodes of these subsystems often carry important information about withdrawals and cannot be removed without losing accuracy. With the GMPA exact accuracy (where the solution is completely identical to the full solution of the hydraulic equations) is maintained while reducing the size of the system to be solved by magnitude.

In this paper the basic idea of the GMPA is used for the derivation of the sensitivity matrix of the global topological minor subgraph. The development is based on the two assumptions that, first, the graph theoretical forest has been removed from the system and, second, that only Demand Driven Analysis (DDA) is considered.

In [3] the topological minor subgraph of a network graph that consists of supernodes and superlinks was introduced. Whereas the set of supernodes is a real subset of the original set of nodes, the superlinks replace the series of original links between the two supernodes. Fig. 1 shows the network graph of an example system (left) and its topological minor (right). The supernodes are the reference node R (by definition) and nodes a and b. The identification of the supernodes is simple for the network core: all nodes with nodal degree > 2. The superlinks consist of the paths between the supernodes. For example, in Fig. 1, superlink s2 replaces the original links 2, 3 and 4. If for each superlink one arbitrary link has to be chosen as so called internal tree chord, the links can be subdivided into the internal tree chord and an arbitrary number of internal tree links.

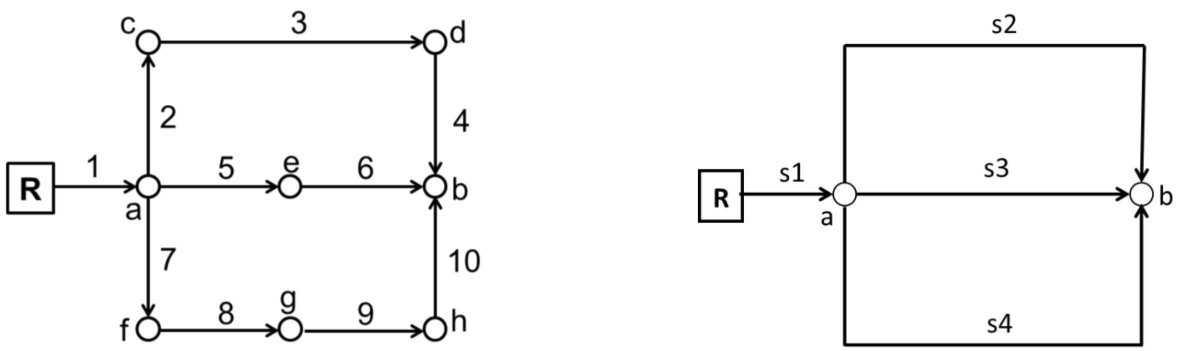

Fig. 1. (a) example system; (b) and its topological minor (supernodes and superlinks).

In [3] it was shown that the incidence matrix of the topological minor subgraph can be calculated analytically based on partitioning of the original incidence matrix of the network graph. For that purpose, in Eq. (1) the columns in the link-node incidence matrix **A** that refer to supernodes are separated from the columns that refer to interior nodes. Similarly, the rows in **A** that refer to internal tree chords are separated from internal tree branches.

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \tag{1}$$

The upper rows $\mathbf{A}_{1x}$ (first index 1) refer to the path tree branches and the lower rows $\mathbf{A}_{2x}$ (first index 2) refer to the internal path tree chords. The second index distinguishes the supernodes from the intermediate nodes (1: supernodes; 2: interior nodes). It can be shown that submatrix $\mathbf{A}_{12}$ is square and invertible and has an analytical inverse. Based on this partitioning the hydraulic steady-state calculation of large complex pipe networks can be split into a local and a global solution [3]. Whereas the global solution includes solving the nonlinear system of equations for the topological minor subgraph that consists of supernodes and superlinks, the local solution for nodes and links in the interior of the partitioned superlinks includes only linear calculations. In [3] it was shown that the matrix $\mathbf{J}_S$ for the smaller topological minor equations has the same structure as the Jacobian for the original system.

$$\mathbf{J}_S = \mathbf{A}_S^T \mathbf{F}_S^{-1} \mathbf{A}_S \tag{2}$$

The index *S* indicates that the matrices refer to the topological subgraph. $\mathbf{J}_S$ is a square and positive definite $n_S \times n_S$ matrix, n_S is the number of supernodes. The $m_S \times n_S$ incidence matrix (m_S: number of superlinks) of the topological subgraph, $\mathbf{A}_S$, can be calculated from the partitioned original incidence matrix [3]:

$$\mathbf{A}_S = \mathbf{A}_{21} - \mathbf{A}_{22} \mathbf{A}_{12}^{-1} \mathbf{A}_{11} \tag{3}$$

The $m_S \times m_S$ diagonal matrix $\mathbf{F}_S$ includes the sum of the derivatives of the headloss equations of links in a superlink and can be also calculated from the partitioned original **F** and **A** matrices:

$$\mathbf{F}_S = \mathbf{F}_2 - \mathbf{A}_{22} \mathbf{A}_{12}^{-1} \mathbf{F}_1^{-1} \mathbf{A}_{12}^{-T} \mathbf{A}_{22}^T \tag{4}$$

In the following it will be shown that the inverse matrix $\mathbf{J}_S$ of Eq. (2) is identical with the sensitivities for the supernodes that are calculated by inversion of the full system Jacobian matrix. The result implies that to get these sensitivities a much smaller system has to be inverted than it was the case for the full system. Often it is sufficient to know the sensitivities of a subset of network nodes and links that have great importance. The supernodes usually represent four-way pipe junctions and T-junctions and therefore are of special importance.

The idea that is central to this paper will first be explained in more detail followed by a numerical example for the network in Fig. 1. The full proof of the equivalence can be found in the Appendix.

2. Sensitivity of the topological submatrix

It is well known [1] that the sensitivity matrix of the nodal heads with respect to the inflows and outflows **Q** at the nodes of a general water supply network graph with incidence matrix **A** and headloss derivative matrix **F** is determined by the inverse Schur Complement matrix

$$\nabla_{\mathbf{Q}} \mathbf{H} = -[\mathbf{A}^T \mathbf{F}^{-1} \mathbf{A}]^{-1} \tag{5}$$

The inverse in Eq. (5) has a central meaning for all kind of sensitivity calculations since all the other sensitivities include it as well. For example, the sensitivity of pipe flows **q** with respect to nodal demands **Q** is:

$$\nabla_{\mathbf{Q}} \mathbf{q} = -\mathbf{F}^{-1} \mathbf{A} [\mathbf{A}^T \mathbf{F}^{-1} \mathbf{A}]^{-1} \tag{6}$$

The Schur Matrix of the partitioned system is:

$$\mathbf{A}^T\mathbf{F}^{-1}\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11}^T\mathbf{F}_1^{-1}\mathbf{A}_{11} + \mathbf{A}_{21}^T\mathbf{F}_2^{-1}\mathbf{A}_{21} & \mathbf{A}_{11}^T\mathbf{F}_1^{-1}\mathbf{A}_{12} + \mathbf{A}_{21}^T\mathbf{F}_2^{-1}\mathbf{A}_{22} \\ \mathbf{A}_{12}^T\mathbf{F}_1^{-1}\mathbf{A}_{11} + \mathbf{A}_{22}^T\mathbf{F}_2^{-1}\mathbf{A}_{21} & \mathbf{A}_{12}^T\mathbf{F}_1^{-1}\mathbf{A}_{12} + \mathbf{A}_{22}^T\mathbf{F}_2^{-1}\mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} \quad (7)$$

The matrix $\mathbf{H}$ is introduced for a more convenient notation in the later equations.

Due to the separation of supernodes and interior nodes, the matrix in Eq. (7) has a certain structure that allows the efficient calculation of the inverse. First the square and symmetric bottom right block $\mathbf{H}_{22}$ is considered. The submatrix consists of independent blocks that refer to the interior nodes of the superlinks. Since the supernodes are not included the submatrix is the same as for a system where all the supernodes are fixed head nodes.

$$\mathbf{H}_{22} = \mathbf{A}_{12}^T\mathbf{F}_1^{-1}\mathbf{A}_{12} + \mathbf{A}_{22}^T\mathbf{F}_2^{-1}\mathbf{A}_{22} = \begin{bmatrix} x & & & & & \\ & x & x & & & \\ & x & x & & & \\ & & & x & x & \\ & & & x & x & x \\ & & & & x & x \end{bmatrix} \quad (8)$$

The matrix $\mathbf{H}_{22}$ is always invertible since it is the sum of a symmetric, positive definite matrix $\mathbf{A}_{12}^T\mathbf{F}_1^{-1}\mathbf{A}_{12}$ first term (since $\mathbf{A}_{12}$ has full rank) and the second matrix $\mathbf{A}_{22}^T\mathbf{F}_2^{-1}\mathbf{A}_{22}$ is diagonal, non-negative definite. A more physical description of the two matrices is as follows: the first matrix is the Schur matrix of the internal tree subnetworks. An analytical inverse exists in this case. The second matrix connects the internal trees with the second supernode by introducing the internal tree chords.

As can be seen from the example in Eq. (8) the submatrix $\mathbf{H}_{22}$ is block tridiagonal, a property which allows a very efficient calculation of the inverse. This property can now be used for the calculation of the Schur complement of the partitioned original Schur matrix for the upper left block of the original matrix:

$$\mathbf{S}^{-1} = [\mathbf{H}_{11} - \mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{12}^T]^{-1} \quad (9)$$

Eq. (9) includes the formula for the calculation of the sensitivities of the heads at the supernodes with respect to the demands at supernodes. Often it suffices to know the sensitivity of a subset of nodes. The supernodes are well suited due to their connectivity property in the network graph. It is worth noting that also the memory requirement is an issue for large networks since the sensitivity matrix is a full matrix for connected systems (no zero elements). If the full matrix is required either the calculation can be extended by application of the full theorem of Schur:

$$[\mathbf{A}^T\mathbf{F}^{-1}\mathbf{A}]^{-1} = \begin{bmatrix} \mathbf{S}^{-1} & -\mathbf{S}^{-1}\mathbf{H}_{12}\mathbf{H}_{22}^{-1} \\ -\mathbf{H}_{22}^{-1}\mathbf{H}_{21}\mathbf{S}^{-1} & \mathbf{H}_{22}^{-1} + \mathbf{H}_{22}^{-1}\mathbf{H}_{21}\mathbf{S}^{-1}\mathbf{H}_{12}\mathbf{H}_{22}^{-1} \end{bmatrix} \quad (10)$$

The application of the theorem of Schur greatly decreases the cost for calculation of the inverse. In what follows an alternative method will be presented that is even more efficient and has an intuitive meaning. Based on Eq. (7) and Eq. (9) the system for calculation of the inverse can be written as

$$\begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Y}^T & \mathbf{Z} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \\ & \mathbf{I} \end{bmatrix} \quad (11)$$

As already mentioned above matrix $\mathbf{H}_{22}$ is the sum of the positive definite matrix $\mathbf{A}_{12}^T\mathbf{F}_1\mathbf{A}_{12}$ and the non-negative definite matrix $\mathbf{A}_{22}^T\mathbf{F}_2^{-1}\mathbf{A}_{22}$ and therefore invertible and we get:

$$\mathbf{X} = (\mathbf{H}_{11} - \mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{12}^{T})^{-1} \tag{12}$$

And therefore

$$[\mathbf{A}^{T}\mathbf{F}^{-1}\mathbf{A}]^{-1} = \begin{bmatrix} \mathbf{X} & x \\ x & x \end{bmatrix} \tag{13}$$

Now we show that the inverse $\mathbf{X}$ can be calculated in a more convenient way by inverting the matrix $\mathbf{J}_S$ of Eq. (2) of the topological minor subgraph. For that purpose we have to show that both calculations are equivalent. Therefore it has to be proven that

$$(\mathbf{H}_{11} - \mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{12}^{T}) = \mathbf{A}_S^{T}\mathbf{F}_S^{-1}\mathbf{A}_S \tag{14}$$

For the full proof the reader is referred to the Appendix. Eq. (14) implies that the sensitivities of the supernode heads with respect to the supernode demands can be simply calculated by the inversion of the matrix $\mathbf{A}_S^{T}\mathbf{F}_S^{-1}\mathbf{A}_S$. Here, the diagonal matrix $\mathbf{F}_S$ includes for every superlink on the main diagonal the sum of headlosses of the links that are replaced by the superlink. The incidence matrix $\mathbf{A}_S$ is the incidence matrix of the network graph that consists of supernodes and superlinks only. The method of the second alternative is even more efficient then the calculation by Eq. (9) since the inversion of the tridiagonal blocks in $\mathbf{H}_{22}$ is not required anymore.

Numerical Example

In the following example the system in Fig. 1 is considered. Table 1 shows the pipe properties length (L), Diameter (D) and roughness (Hazen-Williams coefficient C_{HW}) and the results of a steady-state calculation for the nodal demands Q and reservoir head H at node R that are given in Table 2.

Table 1. Links of network Fig. 1

ID	L [m]	D [mm]	C_{HW}	q [m3/h]	h [m]	$\mathbf{F}_{ii}$ [s/m²]
1	1.000	300	100	360.00	10,45	193,47
2	1.000	200	100	90,68	5,86	430,70
3	1.500	150	100	60,68	16,95	1862,90
4	1.200	200	100	81,01	5,70	469,51
5	800	200	100	178,31	16,40	613,04
6	800	150	100	118,31	31,14	1755,01
7	800	150	100	48,31	5,93	818,24
8	1.200	100	100	31,01	28,20	6062,57
9	1.000	100	100	20,68	11,09	3576,73
10	800	100	100	-31,68	-19,57	4116,67

Table 2. Nodes of network Fig. 1

ID	Q [m3/h]	H [m]
R	-360.00	150,00
a	10.00	139,55
b	20.00	105,65
c	30.00	133,70
d	40.00	116,74
e	50.00	133,85
f	60.00	123,16
g	70.00	92,01
h	80.00	86,08

The matrix $\mathbf{A}^{T}\mathbf{F}^{-1}\mathbf{A}$ for the full system is shown in Fig. 2. The block diagonal submatrix $\mathbf{H}_{22}$ is shown in blue. The top left block refers to superlink S2 (interior nodes c, d), the middle block to S3 (interior node e) and the tridiagonal block bottom right to superlink S4 (interior nodes f, g, h).

	a	b	c	d	e	f	g	h
a	0.01125	0	-0.00232	0	-0.00213	-0.00163	0	0
b	0	0.00069	0	-0.00028	-0.00016	0	0	-0.00024
c	-0.00232	0	0.00286	-0.00054	0	0	0	0
d	0	-0.00028	-0.00054	0.00082	0	0	0	0
e	-0.00213	-0.00016	0	0	0.00229	0	0	0
f	-0.00163	0	0	0	0	0.00220	-0.00057	0
g	0	0	0	0	0	-0.00057	0.00179	-0.00122
h	0	-0.00024	0	0	0	0	-0.00122	0.00147

Fig. 2. Schur Complement of the Jacobian for the full system

The inverse matrix of $\mathbf{A}^T\mathbf{F}^{-1}\mathbf{A}$ for the full system, which is the negative sensitivity matrix $-\nabla_{\mathbf{Q}}\mathbf{H}$, is shown in Fig. 3 below.

	a	b	c	d	e	f	g	h
a	193.47	193.47	193.47	193.47	193.47	193.47	193.47	193.47
b	193.47	2365.64	352.84	1042.16	349.60	375.81	897.82	1141.19
c	193.47	352.84	604.26	518.16	204.93	206.85	245.15	263.01
d	193.47	1042.16	518.16	1922.53	254.47	264.71	468.67	563.76
e	193.47	349.60	204.93	254.47	640.46	206.58	244.10	261.59
f	193.47	375.81	206.85	264.71	206.58	770.36	666.85	618.60
g	193.47	897.82	245.15	468.67	244.10	666.85	2022.05	1835.65
h	193.47	1141.19	263.01	563.76	261.59	618.60	1835.65	2403.07

Fig. 3. Inverse of the Schur Complement of the Jacobian matrix of the full system

Now, the topological subgraph shown on the right in Fig. 1 is considered. The matrix of headloss derivatives $\mathbf{F}_S$ of the superlinks is as follows:

$$\mathbf{F}_S = \begin{pmatrix} \mathbf{F}_{11} & 0 & 0 & 0 \\ 0 & \mathbf{F}_{22}+\mathbf{F}_{33}+\mathbf{F}_{44} & 0 & 0 \\ 0 & 0 & \mathbf{F}_{55}+\mathbf{F}_{66} & 0 \\ 0 & 0 & 0 & \mathbf{F}_{77}+\mathbf{F}_{88}+\mathbf{F}_{99}+\mathbf{F}_{10\,10} \end{pmatrix} = \begin{pmatrix} 193.47 & 0 & 0 & 0 \\ 0 & 5{,}870.33 & 0 & 0 \\ 0 & 0 & 6{,}532.08 & 0 \\ 0 & 0 & 0 & 7{,}302.96 \end{pmatrix}$$

With the incidence matrix $\mathbf{A}_S$ the Schur Complement matrix of the topological subgraph can be calculated:

$$\mathbf{A}_S = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ -1 & 1 \\ -1 & 1 \end{pmatrix}, \mathbf{A}_S^T\mathbf{F}_S^{-1}\mathbf{A}_S = \begin{pmatrix} 0.00563 & -0.00046 \\ -0.00046 & 0.00046 \end{pmatrix}, \nabla_{\mathbf{Q}_S}\mathbf{H}_S = -\left[\mathbf{A}_S^T\mathbf{F}_S^{-1}\mathbf{A}_S\right]^{-1} = \begin{pmatrix} -193.47 & -193.47 \\ -193.47 & -2{,}365.64 \end{pmatrix}$$

Comparison of the first two rows and columns of the full matrix (multiplied by -1) in Fig. 3 with the sensitivity matrix $\nabla_{\mathbf{Q}_S}\mathbf{H}_S = -\left[\mathbf{A}_S^T\mathbf{F}_S^{-1}\mathbf{A}_S\right]^{-1}$ of the topological subgraph shows that the values for the supernodes a and b are identical. For example, the sensitivity of head at node b with respect to a change in demand at the same node is -2,365.64 m/(m3/s). The sensitivity says that the demand change of 10 m3/h (0.002778 m3/s) would lead to a head change of -6.57 m. Please note that the first order sensitivity includes a linearization of the original non-linear system. The correct result in this case would be – 6.74 m.

Conclusions

In this paper an efficient method for calculation of the sensitivities of the supernodes has been proposed. It was shown that the calculation of the inverse Schur Complement for the topological subgraph delivers identical results for these nodes in the same way as for the inverse of the full system. Especially, with consideration of the fact that the inverse matrix has full fill-in (no zero elements) for a connected graph the focus on the supernodes can help to overcome limitations of the full system. If the correct flows for each link are used that might be calculated for example by application of the Graph Matrix Partitioning Algorithm (GMPA) the sensitivities in this case are exactly the same as for the full system. That means that the proposed method includes no simplification by aggregation.

There exists a wide field of application for sensitivities. For instance, being first order approximations they can be included in gradient based parameter optimization algorithms. Other examples are optimal allocation problems such as placement of pressure sensors for monitoring and identification of leakage [4]. The proposed approach has some limitations based on the assumption that the demands are fixed (not pressure dependent) and that the graph theoretical forest is excluded. Current research focuses on the generalization of the method to PDM problems that take into consideration the forest as well.

Acknowledgements

The work presented in this paper is part of the French-German project ResiWater and funded by the French Agence Nationale de la Recherche (ANR; project: ANR-14-PICS-0003) and the German Ministry for Education and Research (BMBF; project: BMBF-13N13690).

Appendix A. Proof of $\left(\mathbf{H}_{11} - \mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{12}^T\right) = \mathbf{A}_S^T\mathbf{F}_S^{-1}\mathbf{A}_S$

The objective is to show the following equivalence:

$$(\mathbf{H}_{11} - \mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{12}^T) = \mathbf{A}_S^T\mathbf{F}_S^{-1}\mathbf{A}_S \tag{A.0}$$

From the previously published paper [3] it is known that:

$$\mathbf{P} = -\mathbf{A}_{22}\mathbf{A}_{12}^{-1} \tag{A.1}$$

$$\mathbf{F}_P = \mathbf{F}_2 + \mathbf{P}\mathbf{F}_1\mathbf{P}^T = \mathbf{F}_2 + \mathbf{A}_{22}\mathbf{A}_{12}^{-1}\mathbf{F}_1\ \mathbf{A}_{12}^{-T}\ \mathbf{A}_{22}^T \tag{A.2}$$

$$\mathbf{A}_P = \mathbf{A}_{21} - \mathbf{A}_{22}\mathbf{A}_{12}^{-1}\mathbf{A}_{11} \tag{A.3}$$

With the abbreviation $\mathbf{V} = \mathbf{A}_{12}^T\mathbf{F}_1^{-1}\mathbf{A}_{12}$ ($\mathbf{V}^{-1} = \mathbf{A}_{12}^{-1}\mathbf{F}_1\ \mathbf{A}_{12}^{-T}$) we have:

$$\begin{aligned}\mathbf{A}_{22}\mathbf{V}^{-1}\mathbf{H}_{22} &= \mathbf{A}_{22}\mathbf{V}^{-1}(\mathbf{V} + \mathbf{A}_{22}^T\mathbf{F}_1^{-1}\mathbf{A}_{22})\\ &= \mathbf{A}_{22} + \mathbf{A}_{22}\mathbf{V}^{-1}\ \mathbf{A}_{22}^T\mathbf{D}_1^{-1}\mathbf{A}_{22}\\ &= \left(\mathbf{I} + \mathbf{A}_{22}\mathbf{V}^{-1}\ \mathbf{A}_{22}^T\mathbf{F}_1^{-1}\right)\mathbf{A}_{22}\\ &= \left(\mathbf{F}_2 + \mathbf{A}_{22}\mathbf{V}^{-1}\ \mathbf{A}_{22}^T\right)\mathbf{F}_2^{-1}\mathbf{A}_{22}\left(\mathbf{F}_2 + \mathbf{A}_{22}\mathbf{A}_{12}^{-1}\mathbf{F}_1\ \mathbf{A}_{12}^{-T}\ \mathbf{A}_{22}^T\right)\mathbf{F}_2^{-1}\mathbf{A}_{22}\\ &= \mathbf{F}_P\mathbf{F}_2^{-1}\mathbf{A}_{22}\end{aligned} \tag{A.4}$$

Thus, the equivalent statements follow:

$$\mathbf{A}_{22}\mathbf{V}^{-1}\mathbf{H}_{22} = \mathbf{F}_S\mathbf{F}_2^{-1}\mathbf{A}_{22} \tag{A.5}$$

$$\mathbf{A}_{22}\mathbf{V}^{-1} = \mathbf{F}_S\mathbf{F}_2^{-1}\mathbf{A}_{22}\mathbf{H}_{22}^{-1} \tag{A.6}$$

$$\mathbf{F}_S^{-1}\mathbf{A}_{22} = \mathbf{F}_2^{-1}\mathbf{A}_{22}\mathbf{H}_{22}^{-1}\mathbf{V} \tag{A.7}$$

$$\mathbf{A}_{22} = \mathbf{F}_S\mathbf{F}_2^{-1}\mathbf{A}_{22}\mathbf{H}_{22}^{-1}\mathbf{V} \tag{A.8}$$

The incidence matrix of the topological minor subgraph can be rewritten from Eq. (6) considering that $A_{12}^{-1} = V^{-1}A_{12}^TD_1^{-1}$:

$$\begin{aligned}\mathbf{A}_S &= \mathbf{A}_{21} - \mathbf{A}_{22}\mathbf{A}_{12}^{-1}\mathbf{A}_{11}\\ &= \mathbf{A}_{21} - \mathbf{A}_{22}\mathbf{V}^{-1}\mathbf{A}_{12}^T\mathbf{F}_1^{-1}\mathbf{A}_{11}\\ &= \mathbf{A}_{21} - \mathbf{A}_{22}\mathbf{V}^{-1}(\mathbf{H}_{12}^T - \mathbf{A}_{22}^T\mathbf{F}_2^{-1}\mathbf{A}_{21})\\ &= \mathbf{A}_{21} - \mathbf{A}_{22}\mathbf{V}^{-1}\mathbf{H}_{12}^T + \mathbf{A}_{22}\mathbf{A}_{12}^{-1}\mathbf{F}_1\ \mathbf{A}_{12}^{-T}\mathbf{A}_{22}^T\mathbf{F}_2^{-1}\mathbf{A}_{21}\\ &= \mathbf{A}_{21} - \mathbf{A}_{22}\mathbf{V}^{-1}\mathbf{H}_{12}^T + (\mathbf{F}_S - \mathbf{F}_2)\mathbf{F}_2^{-1}\mathbf{A}_{21}\\ &= \mathbf{F}_S\mathbf{F}_2^{-1}\mathbf{A}_{21} - \mathbf{A}_{22}\mathbf{V}^{-1}\mathbf{H}_{12}^T\end{aligned} \tag{A.9}$$

Finally, combining Eq. (A.8) and Eq. (A.9) we get

$$\mathbf{F}_S^{-1}\mathbf{A}_S = \mathbf{F}_2^{-1}\mathbf{A}_{21} - \mathbf{F}_2^{-1}\mathbf{A}_{22}\mathbf{H}_{22}^{-1}\mathbf{H}_{12}^T \tag{A.10}$$

By multiplication of Eq. (A.10) with $\mathbf{A}_{21}^T$ and rearranging of the equations the left hand side of Eq. (A.0) can be separated:

$$\mathbf{A}_{21}^T\ \mathbf{F}_S^{-1}\mathbf{A}_S = \mathbf{A}_{21}^T\mathbf{F}_S\mathbf{F}_2^{-1}\mathbf{A}_{21} - \mathbf{A}_{21}^T\mathbf{F}_S\mathbf{F}_2^{-1}\mathbf{A}_{22}\mathbf{H}_{22}^{-1}\mathbf{H}_{12}^T$$

$$\mathbf{A}_{21}^{T}\ \mathbf{F}_{S}^{-1}\mathbf{A}_{S} = (\mathbf{H}_{11} - \mathbf{A}_{11}^{T}\ \mathbf{F}_{1}^{-1}\mathbf{A}_{11}) - (\mathbf{H}_{12} - \mathbf{A}_{11}^{T}\ \mathbf{F}_{1}^{-1}\mathbf{A}_{12})\ \ \mathbf{H}_{22}^{-1}\mathbf{H}_{12}^{T}$$
$$\mathbf{A}_{21}^{T}\mathbf{F}_{S}^{-1}\mathbf{A}_{S} + \mathbf{A}_{11}^{T}\ \mathbf{F}_{1}^{-1}\mathbf{A}_{11} - \mathbf{A}_{11}^{T}\ \mathbf{F}_{1}^{-1}\mathbf{A}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{12}^{T} = \mathbf{H}_{11} - \mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{12}^{T} \qquad \text{(A.11)}$$

If it can be proven that the left hand side of Eq. (A.11) is equivalent with $\mathbf{A}_{S}^{T}\mathbf{D}_{S}^{-1}\mathbf{A}_{S}$ this proves also Eq. (A.0). The following equivalent expressions hold:

$$\mathbf{H}_{12}^{T} - \mathbf{H}_{22}\mathbf{H}_{22}^{-1}\mathbf{H}_{12}^{T} = \mathbf{0}$$
$$\mathbf{A}_{22}^{T}\mathbf{F}_{2}^{-1}\mathbf{A}_{21} + \mathbf{A}_{12}^{T}\mathbf{F}_{1}^{-1}\mathbf{A}_{11} - \left(\mathbf{A}_{22}^{T}\mathbf{F}_{2}^{-1}\mathbf{A}_{22} + \mathbf{A}_{12}^{T}\mathbf{F}_{1}^{-1}\mathbf{A}_{12}\right)\mathbf{H}_{22}^{-1}\mathbf{H}_{12}^{T} = \mathbf{0}$$
$$\mathbf{A}_{12}^{-T}\mathbf{A}_{22}^{T}\mathbf{F}_{2}^{-1}\mathbf{A}_{21} + \mathbf{A}_{12}^{-T}\mathbf{A}_{12}^{T}\mathbf{F}_{1}^{-1}\mathbf{A}_{11} - \mathbf{A}_{12}^{-T}\left(\mathbf{A}_{22}^{T}\mathbf{F}_{2}^{-1}\mathbf{A}_{22} + \mathbf{A}_{12}^{T}\mathbf{F}_{1}^{-1}\mathbf{A}_{12}\right)\mathbf{H}_{22}^{-1}\mathbf{H}_{12}^{T} = \mathbf{0}$$
$$\mathbf{A}_{12}^{-T}\mathbf{A}_{22}^{T}\mathbf{F}_{2}^{-1}\mathbf{A}_{21} + \mathbf{F}_{1}^{-1}\mathbf{A}_{11} - \left(\mathbf{A}_{12}^{-T}\mathbf{A}_{22}^{T}\mathbf{F}_{2}^{-1}\mathbf{A}_{22} + \mathbf{F}_{1}^{-1}\mathbf{A}_{12}\right)\mathbf{H}_{22}^{-1}\mathbf{H}_{12}^{T} = \mathbf{0}$$
$$\mathbf{A}_{11}^{T}\mathbf{A}_{12}^{-T}\mathbf{A}_{22}^{T}\mathbf{F}_{2}^{-1}\mathbf{A}_{21} + \mathbf{A}_{11}^{T}\mathbf{F}_{1}^{-1}\mathbf{A}_{11} - \left(\mathbf{A}_{11}^{T}\mathbf{A}_{12}^{-T}\mathbf{A}_{22}^{T}\mathbf{F}_{2}^{-1}\mathbf{A}_{22} + \mathbf{A}_{11}^{T}\mathbf{F}_{1}^{-1}\mathbf{A}_{12}\right)\mathbf{H}_{22}^{-1}\mathbf{H}_{12}^{T} = \mathbf{0}$$
$$\mathbf{A}_{11}^{T}\mathbf{A}_{12}^{-T}\mathbf{A}_{22}^{T}\mathbf{F}_{2}^{-1}\mathbf{A}_{21} + \mathbf{A}_{11}^{T}\mathbf{F}_{1}^{-1}\mathbf{A}_{12}\mathbf{A}_{12}^{-1}\mathbf{A}_{11} - \left(\mathbf{A}_{11}^{T}\mathbf{A}_{12}^{-T}\mathbf{A}_{22}^{T}\mathbf{F}_{2}^{-1}\mathbf{A}_{22} + \mathbf{A}_{11}^{T}\mathbf{F}_{1}^{-1}\mathbf{A}_{12}\right)\mathbf{H}_{22}^{-1}\mathbf{H}_{12}^{T} = \mathbf{0}$$
$$\mathbf{A}_{11}^{T}\mathbf{A}_{12}^{-T}\mathbf{A}_{22}^{T}\mathbf{F}_{2}^{-1}\mathbf{A}_{21} - \mathbf{A}_{11}^{T}\mathbf{A}_{12}^{-T}\mathbf{A}_{22}^{T}\mathbf{F}_{2}^{-1}\mathbf{A}_{22}\ \mathbf{H}_{22}^{-1}\mathbf{H}_{12}^{T} + \mathbf{A}_{11}^{T}\mathbf{F}_{1}^{-1}\ \mathbf{A}_{12}\left(\mathbf{A}_{12}^{-1}\mathbf{A}_{11} - \mathbf{H}_{22}^{-1}\mathbf{H}_{12}^{T}\right) = \mathbf{0} \qquad \text{(A.12)}$$

Combining Eq. (A.12) and Eq. (A.10) delivers:

$$\mathbf{A}_{11}^{T}\mathbf{A}_{12}^{-T}\mathbf{A}_{22}^{T}\mathbf{F}_{S}^{-1}\mathbf{A}_{S}\ + \mathbf{A}_{11}^{T}\mathbf{F}_{1}^{-1}\ \mathbf{A}_{12}\left(\mathbf{A}_{12}^{-1}\mathbf{A}_{11} - \mathbf{H}_{22}^{-1}\mathbf{H}_{12}^{T}\right) = \mathbf{0}$$
$$\mathbf{A}_{11}^{T}\mathbf{A}_{12}^{-T}\mathbf{A}_{22}^{T}\mathbf{F}_{S}^{-1}\mathbf{A}_{S}\ + \mathbf{A}_{11}^{T}\mathbf{F}_{1}^{-1}\ \mathbf{A}_{11} - \mathbf{A}_{11}^{T}\mathbf{F}_{1}^{-1}\mathbf{A}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{12}^{T} = \mathbf{0}$$
$$(\mathbf{A}_{21}^{T} - \mathbf{A}_{S}^{T})\ \mathbf{F}_{S}^{-1}\mathbf{A}_{S}\ + \mathbf{A}_{11}^{T}\mathbf{F}_{1}^{-1}\ \mathbf{A}_{11} - \mathbf{A}_{11}^{T}\mathbf{F}_{1}^{-1}\mathbf{A}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{12}^{T} = \mathbf{0} \qquad \text{(A.13)}$$

Rearrangement of the left and right side of the equality sign finally gives the desired result:

$$\mathbf{A}_{S}^{T}\ \mathbf{F}_{S}^{-1}\mathbf{A}_{S}\ =\ \mathbf{A}_{11}^{T}\mathbf{F}_{1}^{-1}\ \mathbf{A}_{11} + \mathbf{A}_{21}^{T}\ \mathbf{F}_{S}^{-1}\mathbf{A}_{S}\ - \mathbf{A}_{11}^{T}\mathbf{F}_{1}^{-1}\mathbf{A}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{12}^{T} \qquad \text{(A.14)}$$

□

References

[1] Piller, O., Gilbert, D., and Zyl, J. E. V. (2010). "Dual Calibration for Coupled Flow and Transport Models of Water Distribution Systems." Water Distribution Systems Analysis 2010, E. L. Kevin, Y. C. Christopher, O. Avi, and L. P. Ian, eds., ASCE, 722-731.

[2] Todini, E.; Pilati, S. (1988). “A gradient algorithm for the analysis of pipe networks.” in Computer Applications in Water Supply, Research Studies Press, Letchworth, Hertfordshire, UK.

[3] Deuerlein, J., Elhay, S., and Simpson, A. (2015). "Fast Graph Matrix Partitioning Algorithm for Solving the Water Distribution System Equations." J. Water Resour. Plann. Manage. , 10.1061/(ASCE)WR.1943-5452.0000561 , 04015037.

[4] Pinzinger, R.; Deuerlein, J.; Wolters, A.; Simpson, A. R.: “Alternative Approaches for Solving the Sensor Placement Problem in Large Networks” 13th Annual Symposium on Water Distribution Systems Analysis, WDSA 2011, ASCE, Palm Springs, USA, 22-26 May, 2011.